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# AN ABSTRACT FOR “AN EXPLICIT FORMULA FOR THE GENERIC NUMBER OF DORMANT INDIGENOUS BUNDLES”

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This article is an abstract for the paper [4], entitled “An explicit formula for the generic number of dormant indigenous bundles”. In *loc.cit.*, we consider and answer affirmatively the following question:

*Can one calculate explicitly the number of dormant indigenous bundles on a general curve?*

Here, a dormant indigenous bundle is an integrable  $\mathbb{P}^1$ -bundle on a proper hyperbolic curve of positive characteristic satisfying certain conditions (cf. Definition 0.1, 0.2). Dormant indigenous bundles were introduced and studied in the  $p$ -adic Teichmüller theory developed by S. Mochizuki in the mid-1990’s (cf. [2], [3]). Recently, K. Joshi proposed a conjecture concerning an explicit formula for the degree over the moduli stack of curves of the moduli stack classifying dormant indigenous bundles (cf. Theorem 0.4). In [4], we give a proof of this conjecture of Joshi.

First, let us recall the precise definition of an indigenous bundle as follows:

**Definition 0.1.** (cf. [4], Definition 2.1)

Let  $S$  be a scheme over a field  $k$ ,  $f : X \rightarrow S$  a proper hyperbolic curve over  $S$  of genus  $g$  ( $> 1$ ).

- (i) Let  $\mathcal{P}^\circledast = (\mathcal{P}, \nabla)$  be a pair consisting of a  $\mathrm{PGL}_2$ -torsor  $\mathcal{P}$  on  $X$  and an (integrable)  $S$ -connection  $\nabla$  on  $\mathcal{P}$  (cf. [4], § 1.5). We shall say that  $\mathcal{P}^\circledast$  is an *indigenous bundle* on  $X/S$  if there exists a globally defined section  $\sigma$  of the associated  $\mathbb{P}^1$ -bundle  $\mathbb{P}_{\mathcal{P}}^1 := \mathcal{P} \wedge^{\mathrm{PGL}_2} \mathbb{P}^1$  which has a nowhere vanishing derivative with respect to the connection  $\nabla$ .
- (ii) Let  $\mathcal{P}_1^\circledast = (\mathcal{P}_1, \nabla_1)$ ,  $\mathcal{P}_2^\circledast = (\mathcal{P}_2, \nabla_2)$  be indigenous bundles on  $X/S$ . An *isomorphism* from  $\mathcal{P}_1^\circledast$  to  $\mathcal{P}_2^\circledast$  is an isomorphism  $\mathcal{P}_1 \xrightarrow{\sim} \mathcal{P}_2$  of  $\mathrm{PGL}_2$ -torsors on  $X$  that is compatible with the respective connections.

Let  $\mathcal{M}_{g,k}$  be the moduli stack of proper hyperbolic curves of genus  $g$  ( $> 1$ ) over  $k$ , and  $(Sch)_{\mathcal{M}_{g,k}}$  the category of  $\mathcal{M}_{g,k}$ -schemes. Denote by

$$\mathcal{S}_{g,k} : (Sch)_{\mathcal{M}_{g,k}} \rightarrow (Set)$$

the set-valued functor on  $(Sch)_{\mathcal{M}_{g,k}}$  which, to any  $\mathcal{M}_{g,k}$ -scheme  $T$ , classifying a curve  $Y/T$ , assigns the set of isomorphism classes of indigenous bundles on  $Y/T$ . It is known (cf. [4], Theorem 3.3) that the functor  $\mathcal{S}_{g,k}$  may be represented by a relative affine space over  $\mathcal{M}_{g,k}$  of relative dimension  $3g - 3$ .

The notion of an indigenous bundle was originally introduced and studied by Gunning in the context of compact hyperbolic Riemann surfaces, i.e., the case where  $k = \mathbb{C}$  (i.e., the complex number field). One may think of an indigenous bundle on a given compact hyperbolic Riemann surface  $X$  as an *algebraic* object that encodes the (*analytic*, i.e., *non-algebraic*) uniformization data for  $X$ . It may be interpreted as a projective structure, i.e., a maximal atlas covered by coordinate charts on  $X$  such that the transition functions may be expressed as Möbius transformations. Also, various equivalent mathematical objects, including certain kinds of differential operators or kernel functions, have been studied by many mathematicians.

In the paper [4], we focus on indigenous bundles (and the moduli space classifying indigenous bundles) in *positive characteristic*, i.e., the case where  $k = \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ . Various properties of such objects were first discussed in the context of the  $p$ -adic Teichmüller theory developed by S. Mochizuki (cf. [2], [3]). One of the key ingredients in the development of this theory is the study of the  $p$ -curvature of indigenous bundles in characteristic  $p$ . The  $p$ -curvature of a connection may be thought of as the obstruction to the compatibility of  $p$ -power structures that appear in certain associated spaces of infinitesimal (i.e., “Lie”) symmetries. The condition that the  $p$ -curvature vanishes identically implies, in particular, the existence of “sufficiently many” horizontal sections locally in the Zariski topology.

Here, we shall recall the definition of the  $p$ -curvature map. Let us fix an algebraic group  $G$  over  $k$  and denote by  $\mathfrak{g}$  the Lie algebra of  $G$ . Let  $(\pi : \mathcal{E} \rightarrow X, \nabla)$  be a pair consisting of a  $G$ -torsor  $\mathcal{E}$  on  $X$  and an  $S$ -connection  $\nabla : \mathcal{T}_{X/S} \rightarrow \tilde{\mathcal{T}}_{\mathcal{E}/S}$  on  $\mathcal{E}$ , i.e., a section of the natural quotient  $\alpha_{\mathcal{E}} : (\pi_* \mathcal{T}_{\mathcal{E}/S})^G =: \tilde{\mathcal{T}}_{\mathcal{E}/S} \rightarrow \mathcal{T}_{X/S}$  (cf. [4], §1.5). If  $\partial$  is a derivation corresponding to a local section  $\partial$  of  $\mathcal{T}_{X/S}$  (respectively,  $\tilde{\mathcal{T}}_{\mathcal{E}} := (\pi_* \mathcal{T}_{\mathcal{E}/S})^G$ ), then we shall denote by  $\partial^{[p]}$  the  $p$ -th iterate of  $\partial$ , which is also a derivation corresponding to a local section of  $\mathcal{T}_{X/S}$  (respectively,  $\tilde{\mathcal{T}}_{\mathcal{E}}$ ). Since  $\alpha_{\mathcal{E}}(\partial^{[p]}) = (\alpha_{\mathcal{E}}(\partial))^{[p]}$  for any local section of  $\mathcal{T}_{X/S}$ , the image of the  $p$ -linear map from  $\mathcal{T}_{X/S}$  to  $\tilde{\mathcal{T}}_{\mathcal{E}/S}$  defined by assigning  $\partial \mapsto \nabla(\partial^{[p]}) - (\nabla(\partial))^{[p]}$  is contained in  $\mathcal{E} \wedge^G \mathfrak{g}$  ( $= \ker(\alpha_{\mathcal{E}})$ ). Thus, we obtain an  $\mathcal{O}_X$ -linear morphism

$$\psi_{(\mathcal{E}, \nabla)} : \mathcal{T}_{X/S}^{\otimes p} \rightarrow \mathcal{E} \wedge^G \mathfrak{g}$$

determined by assigning

$$\partial^{\otimes p} \mapsto \nabla(\partial^{[p]}) - (\nabla(\partial))^{[p]}.$$

We shall refer to the morphism  $\psi_{(\mathcal{E}, \nabla)}$  as the  $p$ -curvature map of  $(\mathcal{E}, \nabla)$ .

**Definition 0.2.** (cf. [4], Definition 3.1)

We shall say that an indigenous bundle  $(\mathcal{P}, \nabla)$  on  $X/S$  is *dormant* if the  $p$ -curvature map of  $(\mathcal{P}, \nabla)$  vanishes identically on  $X$ .

We shall denote by

$$\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz...}}$$

the subfunctor of  $\mathcal{S}_{g, \mathbb{F}_p}$  classifying the set of isomorphism classes of dormant indigenous bundles (cf. the notation “Zzz...”!). S. Mochizuki showed the following properties concerning  $\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots}$ :

**Theorem 0.3.** (cf. [4], Theorem 3.3)

*The functor  $\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots}$  is a closed substack of  $\mathcal{S}_{g, \mathbb{F}_p}$  and may be represented by a smooth, geometrically irreducible Deligne-Mumford stack over  $\mathbb{F}_p$ . Moreover, the natural projection  $\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots} \rightarrow \mathcal{M}_{g, \mathbb{F}_p}$  is finite, faithfully flat, and generically étale.*

In particular, it follows that it makes sense to speak of the *degree*

$$\deg_{\mathcal{M}_{g, \mathbb{F}_p}}(\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots})$$

of  $\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots}$  over  $\mathcal{M}_{g, \mathbb{F}_p}$ . The generic étaleness of  $\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots}$  over  $\mathcal{M}_{g, \mathbb{F}_p}$  implies that if  $X$  is a sufficiently general proper hyperbolic curve of genus  $g$  over an algebraically closed field of characteristic  $p$ , then the number of dormant indigenous bundles on  $X$  is exactly  $\deg_{\mathcal{M}_{g, \mathbb{F}_p}}(\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots})$ . Thus, as we explained above, our main interest is the *explicit computation* of the value  $\deg_{\mathcal{M}_{g, \mathbb{F}_p}}(\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots})$ . The main result of the paper [4] is the following:

**Theorem 0.4.** (cf. [4], Corollary 5.4)

*Suppose that  $p > 2(g-1)$  ( $\iff \frac{1}{2} \cdot p + 1 > g$ ). Then the degree  $\deg_{\mathcal{M}_{g, \mathbb{F}_p}}(\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots})$  of  $\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots}$  over  $\mathcal{M}_{g, \mathbb{F}_p}$  is given by the following formula:*

$$\begin{aligned} \deg_{\mathcal{M}_{g, \mathbb{F}_p}}(\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots}) &= \frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{\theta=1}^{p-1} \frac{1}{\sin^{2g-2}\left(\frac{\pi \cdot \theta}{p}\right)} \\ &\left( = \frac{(-1)^{g-1} \cdot p^{g-1}}{2} \cdot \sum_{\zeta^p=1, \zeta \neq 1} \frac{\zeta^{g-1}}{(\zeta-1)^{2g-2}} \right). \end{aligned}$$

In the case of  $g = 2$ , S. Mochizuki, H. Lange-C. Pauly, and B. Osserman verified (by applying different methods) the equality

$$\deg_{\mathcal{M}_{2, \mathbb{F}_p}}(\mathcal{M}_{2, \mathbb{F}_p}^{\text{Zzz}\dots}) = \frac{1}{24} \cdot (p^3 - p),$$

which is consistent with the assertion in Theorem 0.4. For general  $g$ , K. Joshi conjectured, with his amazing insight, an explicit description, as asserted in Theorem 0.4, of the value  $\deg_{\mathcal{M}_{g, \mathbb{F}_p}}(\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots})$ . (In fact, Joshi has proposed a somewhat more general conjecture. In [4], however, we shall restrict our attention to a certain special case of this more general conjecture.)

Our discussion in [4] follows, to a substantial extent, the ideas discussed in [1], as well as in personal communication to the author by K. Joshi. Indeed, certain of the results obtained in [4] are mild generalizations of the results obtained in [1] concerning rank 2 opers to the case of *families of curves* over quite general base schemes. (Such *relative* formulations are necessary in the theory of [4] in

order to consider *deformations* of various types of data.) Moreover, the insight concerning the connection with the formula of Holla (cf. [4], Theorem 5.1), which is a special case of the Vafa-Intriligator formula, is due to Joshi.

On the other hand, the new ideas introduced in [4] may be summarized as follows. The key technical achievement of [4] lies in the verification of the *vanishing of obstructions* to deformation to characteristic zero of a certain Quot-scheme that is related to  $\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots}$ . (cf. [4], Proposition 4.3, [4], Lemma 4.4, and the discussion in the proof of [4], Lemma 5.2). This vanishing of obstructions is shown by combining the *generic étaleness* of Theorem 0.3 with a certain computation involving the *Riemann-Roch theorem* on the curves under consideration. We then relate the value  $\deg_{\mathcal{M}_{g, \mathbb{F}_p}}(\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots})$  to the degree of the result of base-changing this Quot-scheme to  $\mathbb{C}$  by applying the formula of Holla *directly*.

Finally, we make a remark concerning Theorem 0.4. It is known (cf. [4], § 6.1) that a dormant indigenous bundle corresponds, in a certain sense, to a certain type of rank 2 semistable vector bundle whose pull-back by Frobenius is unstable. Such semistable vector bundles have been studied in a different context. Also, it follows from work of S. Mochizuki, F. Liu and B. Osserman (cf. [4], § 6.1) that there is a bijective correspondence between the lattice points inside a certain convex polytope arising from a 3-regular graph and the set of isomorphism classes of dormant indigenous bundles on a certain (singular) curve (cf. [5], Theorem B). (F. Liu and B. Osserman have shown that the value  $\deg_{\mathcal{M}_{g, \mathbb{F}_p}}(\mathcal{M}_{g, \mathbb{F}_p}^{\text{Zzz}\dots})$  may be expressed as a polynomial with respect to the characteristic of the base field. This result, which may be thought of as a weak version of Theorem 0.4, was obtained by applying Ehrhart's theory concerning the cardinality of the set of lattice points inside a polytope.) By applying such correspondences, the main result of [4] yields explicit computations of the cardinality of such sets of lattice points (cf. [5], Theorem A) and hence of corresponding sets of isomorphism classes of rank 2 semistable vector bundles satisfying certain conditions.

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